Generalized functions - HW 3

March 7, 2016

Question 1

a. Assume first that $g(x) \equiv 1$. We will prove the claim by induction on k. The case $k = 0$ is immediate. Denote $I = \{v_1, ..., v_k\}$ and $I' = I \setminus \{v_k\}$. We note that by the chain rule

$$\left(\frac{\partial^k}{\partial_{v_I}}\tilde{f}\right)(0) = \left(\frac{\partial^{k-1}}{\partial_{v_{I'}}}\frac{\partial}{\partial_{v_k}}f \circ \varphi\right)(0) = \left(\frac{\partial^{k-1}}{\partial_{v_{I'}}}\frac{\partial}{\partial_{v_k}}f \circ \varphi\right)(0) = \left(\frac{\partial^{k-1}}{\partial_{v_{I'}}}\langle \nabla f, \frac{\partial \varphi}{\partial_{v_k}}\rangle\right)(0)$$

But now all first $k-1$ derivatives of $\langle \nabla f, \frac{\partial \varphi}{\partial_{v_k}}\rangle$ must vanish at 0 thus by the induction hypothesis

$$\left(\frac{\partial^{k-1}}{\partial_{v_{I'}}}\langle \nabla f, \frac{\partial \varphi}{\partial_{v_k}}\rangle\right)(0) = \left(\frac{\partial^{k-1}}{\partial D\varphi_{v_{I'}}}\langle \nabla f, \frac{\partial \varphi}{\partial_{v_k}}\rangle\right)(0) = \left(\frac{\partial^{k-1}}{\partial D\varphi_{v_{I'}}}\frac{\partial}{\partial D\varphi_{v_k}}f\right)(0)$$

As desired.
For general g we note that for every multi index J, $|J| \leq k$ we have that $\frac{\partial^{|J|}}{\partial_{v_J}}f \circ \varphi(0) = 0$. We may now combine this fact with the generalized Leibniz law to obtain

$$\left(\frac{\partial^k}{\partial_{v_I}}\tilde{f}\right)(0) = \sum_{J \subset I}\binom{k}{|J|}\frac{\partial^{|J|}}{\partial_{v_J}}f \circ \varphi(0) \cdot \frac{\partial^{k-|J|}}{\partial_{v_{I \setminus J}}}g(0) = \frac{\partial^k}{\partial_{v_I}}f \circ \varphi(0) \cdot g(0)$$

Combining both parts concludes the proof.

b. Consider $f(x) = e^x$ and $\varphi(x) = x^3 + x$. Clearly $f'(0) \neq 0$ and we have that

$$\left(\frac{\partial^2}{\partial_x^2}f \circ \varphi\right)(0) = [e^{x^3+x}]''(0) = [e^{x^3+x}(9x^4 + 6x^2 + 6x + 1)](0) = 1$$

while

$$\frac{\partial^2}{\partial^2 D\varphi_x}f(0) = \frac{\partial}{\partial D\varphi_x}(f' \cdot \varphi')(0) = \frac{\partial}{\partial D\varphi_x}(e^x(3x^2+1))(0) = (e^x(3x^2+1))'(3x^2+1)(0) \neq 1$$

Question 2

If $\varphi^* : C^\infty(Y) \to C^\infty(X)$ is defined as $\varphi^*(f) = f \circ \varphi$, then for we will show that $\mathrm{supp}(\varphi^* f) \subset \varphi^{-1}\mathrm{supp}(f)$. Indeed let $x \in \mathrm{supp}(\varphi^* f)$, then $f(\varphi(x)) \neq 0$ which implies $\phi(x) \in \mathrm{supp}(f)$ or $x \in \varphi^{-1}\mathrm{supp}(f)$. It may be worthwhile to notice that this works only when x is in the interior of $\mathrm{supp}(\varphi^* f)$ but extending it to the boundary is immediate.

Now, let $f \in C_c^\infty(Y)$ then $\mathrm{supp}(f)$ is compact and for proper φ so is $\varphi^{-1}\mathrm{supp}(f)$. Thus, $\mathrm{supp}(\varphi^* f)$ is compact as well as a closed subset of $\varphi^{-1}\mathrm{supp}(f)$. And $\varphi^*(f) \in C_c^\infty(X)$.

Question 3

a. Let $\bar{g} \in \mu_c^\infty(X)$ we will show that $\varphi_*(\bar{g}) \in \mu_c^\infty(Y)$. The fact that $\varphi_*(\bar{g})$ is compactly supported is implied by the next item. Thus we will contend ourselves with showing that it is smooth. Smoothness is a local property, and for every $y \in Y$ we can then find a neighborhood V such that $\varphi|_{\varphi^{-1}(V)}$ is a projection from F^{n+k} to F^n. On this neighborhood we may write $\bar{g}$ as $g \cdot d\lambda$ where λ denotes the Haar measure. It is now the case that $\varphi_*(\bar{g})$ is defined by integration along the fibers of φ, thus if dx, dy are the respective Haar measures on F^k and F^n

$$\varphi_*(\bar{g}) = \left(\int\limits_{F^k} g(x,y)dx \right) dy$$

By denoting $h(y) := \int\limits_{F^n} g(x,y)dx$ we may write $\varphi_*(\bar{g}) = h \cdot dy \in \mu_c^\infty(F^n)$ as promised.

b. Let $\xi \in \mathrm{Dist}_c(X)$ such that $\mathrm{supp}(\xi) = K$. For $f \in C_c^\infty(Y)$ by writing

$$\langle \varphi_*(\xi), f \rangle = \langle \xi, f \circ \phi \rangle$$

We may see that $\varphi_*(\xi)$ must be supported on $\varphi(K)$. Since the continuous image of a compact set must be compact the claim follows. (Hope this is not utter nonsense)

Question 4

We will show that for every $\bar{g} \in \mu_c^\infty(X)$, $\langle \varphi^*(f), \bar{g} \rangle = \langle f \circ \varphi, \bar{g} \rangle$. To be more specific, for every $x \in X$ we will show that there exists a neighborhood in which the above equality holds, since the measures form a sheaf, the claim will follow.

Let $x \in X$, and let V be a neighborhood of y such that $V \simeq F^{n+k}$ and φ acts as a projection from V (this is possible by φ being a submersion). Thus, we may assume WLOG that $\varphi(x_1, ..., x_n, ..., x_{n+k}) = (x_1, ..., x_n)$. But now, by definition, $\langle \varphi^*(f), \bar{g} \rangle = \langle f, \varphi_*(\bar{g}) \rangle$ where $\varphi_*(g)$ is given by integration along the fibers of φ and $\bar{g}$ can be written as $g \cdot d\lambda$, where λ is the Haar measure on F. This the amounts to

$$\varphi_*(\bar{g})(y) = \int\limits_{F^k} g(y,x)dx, \quad x \in F^k, y \in F^n$$

and, finally, by using Fubini's theorem

$$\begin{aligned}
\langle \varphi^*(f), \bar{g} \rangle &= \int\limits_{F^n} f(y) \left(\int\limits_{F^k} g(x,y)dx \right) dy \\
&= \int\limits_{F^k} \int\limits_{F^n} g(y,x) f(y) dy dx = \int\limits_{F^k} \int\limits_{F^n} g(y,x) f \circ \varphi(y,x) dy dx \\
&= \langle f \circ \varphi, \bar{g} \rangle
\end{aligned}$$

Which concludes the proof.

Question 5

We will first show that every $\chi \in \mathbb{R}^\vee$ is of the form $\chi(x) = e^{isx}$ for some $t \in \mathbb{R}$. Indeed, let χ be a non-trivial character. Since χ is a continuous homomorphism its forms a subgroup of the circle which contains an entire interval. In which case it is easy to verify that χ is unto. The isomorphism theorem then shows that χ must have a non-trivial kernel. Let $0 \neq T \in \ker(\chi)$, so for every x, $\chi(x+T) = \chi(x)\chi(T) = \chi(x)$, and χ is periodic.

As a non constant function χ has a smallest period, denoted as T'. The considerations given above display that for every $x \in (0, T')$, $\chi(x) \neq 0$. In particular, there exists $n \in \mathbb{N}$ such that for $x \in (0, \frac{1}{2^n})$, the imaginary part of $\chi(x)$ has constant sign. WLOG the sign is positive (otherwise, we transfer to χ^{-1}).

Let s be minimal such that $\chi(\frac{1}{2^n}) = e^{is}$. Now, for every $k \in \mathbb{N}$, $\chi(\frac{1}{2^n 2^k})$ must equal $e^{i\frac{s}{2^k}}$. And for every $j \leq 2^k$, $\chi(\frac{j}{2^n 2^k}) = e^{i\frac{sj}{2^k}}$. Since $\{\frac{j}{2^n 2^k}\}$ is dense in $[0, \frac{1}{2^n}]$,χ must equal e^{is2^n} there. By rotational invariance, $\chi(x) = e^{is2^n x}$ everywhere.

But now, in the compact open topology a sequence f_n converges to a function f if and only if f_n converges uniformly on every compact subset. It is routine to check that a sequence $e^{it_n x}$ will converge to a function e^{itx} uniformly on every compact subset if and only if $t_n \to t$. Thus the topology on $\mathbb{R}^\vee$ is equivalent to the topology on $\mathbb{R}$ and $\mathbb{R} \approx \mathbb{R}^\vee$